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SUCCESSIVE REMAINDERS OF THE NEWTON SERIES

BY

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ABSTRACT. If f is analytic in the open unit disc D and λ is a sequence of points in D converging to 0, then f admits the Newton series expansion $f(z) = f(\lambda_1) + \Sigma_{n=1}^{\infty} \Delta_\lambda^n f(\lambda_{n+1})(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)$, where $\Delta_\lambda^n f(z)$ is the nth divided difference of f with respect to the sequence λ. The Newton series reduces to the Maclaurin series in case $\lambda_n \equiv 0$. The present paper investigates relationships between the behavior of zeros of the normalized remainders $\Delta_\lambda^k f(z) = \Delta_\lambda^k f(\lambda_{k+1}) + \Sigma_{n=k+1}^{\infty} \Delta_\lambda^n f(\lambda_{n+1})(z-\lambda_{k+1})\cdots(z-\lambda_n)$ of the Newton series and zeros of the normalized remainders $\Sigma_{n=k}^{\infty} a_n z^{n-k}$ of the Maclaurin series for f. Let C_λ be the supremum of numbers $c > 0$ such that if f is analytic in D and each of $\Delta_\lambda^k f(z)$, $0 \le k < \infty$, has a zero in $|z| \le c$, then $f \equiv 0$. The corresponding constant for the Maclaurin series (C_λ, where $\lambda_n \equiv 0$) is called the Whittaker constant for remainders and is denoted by W. We prove that $C_\lambda \ge W$, for all λ, and, moreover, $C_\lambda = W$ if $\lambda \in l_1$. In obtaining this result, we prove that functions f analytic in D have expansions of the form $f(z) = \Sigma_{n=0}^{\infty} \Delta_\lambda^n f(z_n) C_n(z)$, where $|z_n| \le W$, for all n, and $C_n(z)$ is a polynomial of degree n determined by the conditions $\Delta_\lambda^j C_k(z_j) = \delta_{jk}$.

1. **Introduction.** Let f be analytic in the open unit disc D and let λ denote a sequence of points in D. The *Newton series* for f is given by

$$f(z) = f(\lambda_1) + \sum_{n=1}^{\infty} \Delta_\lambda^n f(\lambda_{n+1})(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n), \tag{1.1}$$

where $\Delta_\lambda^n f$ denotes the nth divided difference of f with respect to λ:

$$\Delta_\lambda^0 f(z) = f(z), \quad \Delta_\lambda^n f(z) = \frac{\Delta_\lambda^{n-1} f(z) - \Delta_\lambda^{n-1} f(\lambda_n)}{z-\lambda_n}, \quad n = 1, 2, 3, \cdots. \tag{1.2}$$

The series in (1.1) converges uniformly to f on compact subsets of D for each $\lambda \in c_0$, the space of complex sequences converging to 0 ([10], [4]). Note that (1.1) reduces to the Maclaurin series for f in case $\lambda_n \equiv 0$. If λ is the constant sequence $\lambda_n \equiv z_0$, then (1.1) is simply the Taylor series for f, expanded about z_0. In this case, convergence is guaranteed only in $|z - z_0| < 1 - |z_0|$.

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In the present paper, we consider the behavior of zeros of the successive normalized remainders $\Delta_\lambda^k f(\lambda_{k+1}) + \sum_{n=k+1}^{\infty} \Delta_\lambda^n f(\lambda_{n+1})(z-\lambda_{k+1}) \cdots (z-\lambda_n)$, $0 \le k < \infty$, of the Newton series. In view of the identity

$$\Delta_\lambda^k f(z) = \Delta_\lambda^k f(\lambda_{k+1}) + \sum_{n=k+1}^{\infty} \Delta_\lambda^n f(\lambda_{n+1})(z-\lambda_{k+1}) \cdots (z-\lambda_n), \tag{1.3}$$

which follows from a simple induction argument, our problem is equivalent to studying the zeros of the successive divided differences $\Delta_\lambda^k f$ of f. For $\lambda \in c_0$, let C_λ denote the supremum of positive numbers c such that if f is analytic in D and each of $\Delta_\lambda^k f(z)$, $0 \le k < \infty$, has a zero in $|z| \le c$, then $f \equiv 0$. The bound $C_\lambda \le 1$ is part of the definition. In 1965, M. Pommiez [10] proved that $C_\lambda \ge .536$ for each $\lambda \in c_0$. Pommiez noted that C_λ might be independent of λ for suitably restricted sequences in c_0.

In the case when λ is the *null sequence* ν ($\nu_n \equiv 0$), C_ν has been determined ([1], [2], [3], [6]). Here, one considers the successive normalized remainders $\mathcal{S}^k f(z) = \sum_{n=k}^{\infty} f^{(n)}(0) z^{n-k}/n!$ ($k = 0, 1, 2, \cdots$) of the Maclaurin series, or equivalently, the zeros of the *shift operator* $\mathcal{S}^k f$. The constant $W(\mathcal{S})$ is the supremum of positive numbers c such that if f is analytic in D and each of $\mathcal{S}^k f(z)$, $0 \le k < \infty$, has a zero in $|z| \le c$, then $f \equiv 0$. Clearly, we have

$$C_\nu = W(\mathcal{S}); \tag{1.4}$$

$W(\mathcal{S})$ is known as the *Whittaker constant belonging to* $\mathcal{S}$, and satisfies $.549 < W(\mathcal{S}) < .562$. The following theorem, due to J. D. Buckholtz and J. L. Frank ([3], [2]) completely characterizes $W(\mathcal{S})$.

Theorem. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have radius of convergence* 1 *and let $\epsilon > 0$. Then*

(i) *infinitely many of the partial sums $\sum_{n=0}^{k} a_n z^n$ have all their zeros in the disc $|z| \le W(\mathcal{S})^{-1} + \epsilon$,*

(ii) *infinitely many of the remainders $\sum_{n=k}^{\infty} a_n z^{n-k}$ have no zero in $|z| \le (W(\mathcal{S})^{-1} + \epsilon)^{-1}$,*

(iii) *$W(\mathcal{S})$ cannot be replaced by a larger number in either* (i) *or* (ii).

Because of this result, the number $P = W(\mathcal{S})^{-1}$ is called the *power series constant.*

Following our remarks concerning the null sequence, it is natural to ask whether some extension of (1.4) holds for the nontrivial sequences in c_0. In this direction, our principal result is

Theorem 1. *For each $\lambda \in c_0$, $C_\lambda \ge W(\mathcal{S})$; moreover, $C_\lambda = W(\mathcal{S})$ for each $\lambda \in l_1 = \{\mu : \sum |\mu_n| < \infty\}$.*

This is proved in § 3. To simplify notation in Theorem 1 and its proof, we will abbreviate $W(\mathcal{S})$ to W and drop the subscript λ from Δ_λ^n when no confusion is likely as to the particular sequence λ under consideration.

A further characterization of W, and one which we will need, is obtained from the *remainder polynomials* $B_n(z; z_0, z_1, z_2, \cdots, z_{n-1})$. These are defined inductively by

$$B_0(z) = 1,$$

$$(1.5)\qquad B_n(z; z_0, z_1, \cdots, z_{n-1}) = z^n - \sum_{k=0}^{n-1} z_k^{n-k} B_k(z; z_0, z_1, \cdots, z_{k-1}),$$

for $n = 1, 2, 3, \cdots$, where $\{z_k\}_{k=0}^\infty$ is a sequence of complex numbers. Let $H_n = \max |B_n(0; w_0, w_1, \cdots, w_{n-1})|$, where the maximum is taken over all sequences $\{w_k\}_{k=0}^{n-1}$ in $\bar{D}$. Buckholtz [1] proved that

$$(1.6)\qquad W^{-1} = \lim_{n\to\infty} H_n^{1/n} = \sup_{1\le n<\infty} H_n^{1/n},$$

and that there exists a constant β, $0 < \beta < 1$, such that

$$(1.7)\qquad W^n H_n \ge \beta$$

for $0 \le n < \infty$ [3]. From (1.6) the numerical value of W can be (theoretically) calculated to any desired accuracy.

2. **Preliminaries.** The bound $C_\lambda \ge .536$ is a consequence of the following expansion theorem of Pommiez.

Theorem. *Suppose that f is analytic in D and c is a number such that $0 \le c \le .536$. Let $\{z_n\}_{n=0}^\infty$ be a sequence of complex numbers in $|z| \le c$ and let $\lambda \in c_0$. Then there exists a sequence $\{C_n\}$ of polynomials, C_n of degree n, such that for all $z \in D$,*

$$(2.1)\qquad f(z) = f(z_0) + \sum_{n=1}^\infty \Delta_\lambda^n f(z_n) C_n(z).$$

From (2.1) the conditions $\Delta_\lambda^n f(z_n) = 0$, $0 \le n < \infty$, imply $f \equiv 0$. Thus $C_\lambda \ge 536$, $\lambda \in c_0$.

We seek an expansion of the form (2.1) with milder restrictions on the sequence $\{z_k\}$. Applying (2.1) to $F(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$ and noting $\Delta^k F(z) = (z - \lambda_{k+1})(z - \lambda_{k+2}) \cdots (z - \lambda_n)$, $0 \le k \le n - 1$, and $\Delta^n F(z) = 1$, it follows that the polynomials $C_n(z)$ must satisfy

$$(2.2)\qquad \begin{aligned}(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n) &= (z_0 - \lambda_1)(z_0 - \lambda_2) \cdots (z_0 - \lambda_n) \\ &+ \sum_{k=1}^{n-1} (z_k - \lambda_{k+1}) \cdots (z_k - \lambda_n) C_k(z) + C_n(z).\end{aligned}$$

Then clearly $C_n(z)$ depends on $z_0, z_1, \cdots, z_{n-1}$ and $\lambda_1, \lambda_2, \cdots, \lambda_n$. Taking $C_0(z) = 1$, (2.2) becomes

$$C_n(z; z_0, z_1, \cdots, z_{n-1}; \lambda_1, \lambda_2, \cdots, \lambda_n)$$

$$= (z - \lambda_1) \cdots (z - \lambda_n) - \sum_{k=0}^{n-1} (z_k - \lambda_{k+1}) \cdots (z_k - \lambda_n) \cdot C_k(z; z_0, z_1, \cdots, z_{k-1}; \lambda_1, \lambda_2, \cdots, \lambda_k). \tag{2.3}$$

We therefore take (2.3) as our recursive defining relation, without regard to the restrictions on $\{z_k\}_{k=0}^{n-1}$ imposed by Pommiez's theorem.

Lemma 1. *Let* $\{z_j\}_{j=0}^{\infty}$ *and* $\{\lambda_j\}_{j=1}^{\infty}$ *be sequences of complex numbers. The following identities hold:*

$$\Delta^k C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) = C_{n-k}(z; z_k, \cdots, z_{n-1}; \lambda_{k+1}, \cdots, \lambda_n), \quad \textit{for } 0 \le k \le n, \tag{2.4}$$

$$C_n(z_0; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) = 0, \qquad n \ge 1, \tag{2.5}$$

$$\Delta^k C_n(z_k; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) = \delta_{kn}, \qquad 0 \le k,\ n < \infty, \tag{2.6}$$

where δ_{kn} *denotes the Kronecker delta,*

$$C_n(\alpha z; \alpha z_0, \cdots, \alpha z_{n-1}; \alpha\lambda_1, \cdots, \alpha\lambda_n) = \alpha^n C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n), \qquad n \ge 0, \tag{2.7}$$

where α *is a complex number.*

$$C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) = \sum_{k=0}^{n} C_{n-k}(\lambda_{k+1}; z_k, \cdots, z_{n-1}; \lambda_{k+1}, \cdots, \lambda_n)(z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_k), \quad n \ge 0, \tag{2.8}$$

$$C_n(z; z_0, \cdots, z_{n-1}; 0, \cdots, 0) = B_n(z; z_0, \cdots, z_{n-1}), \qquad n \ge 0. \tag{2.9}$$

Proof. In (2.8), take $(z - \lambda_1) \cdots (z - \lambda_k)$ to be 1 if $k = 0$. Note that (2.6) implies that the polynomials $C_n(z)$, $0 \le n < \infty$, together with the sequence of linear functionals $f \to \Delta^k f(z_k)$, $0 \le k < \infty$, form a biorthonormal system [4].

We establish (2.4) by induction on n. If $n = 0$, then $k = 0$ and the result is trivial. Let m be a positive integer and suppose that for each j such that $0 \le j \le m - 1$ we have $\Delta^k C_j(z; z_0, \cdots, z_{j-1}; \lambda_1, \cdots, \lambda_j) = C_{j-k}(z; z_k, \cdots, z_{j-1}; \lambda_{k+1}, \cdots, \lambda_j)$,

$0 \le k \le j$. Note that $k > j$ implies $\Delta^k C_j(z) = 0$, since $C_j(z)$ is a polynomial of degree j in z. By (2.3) and the induction hypothesis, $k \le m-1$ implies

$$\begin{aligned}
\Delta^k C_m(z; z_0, \cdots, z_{m-1}; \lambda_1, \cdots, \lambda_m) \\
&= \Delta^k (z-\lambda_1)\cdots(z-\lambda_m) \\
&\quad - \sum_{j=0}^{m-1} (z_j - \lambda_{j+1})\cdots(z_j-\lambda_m)\Delta^k C_j(z; z_0, \cdots, z_{j-1}; \lambda_1, \cdots, \lambda_j) \\
&= (z-\lambda_{k+1})\cdots(z-\lambda_m) \\
&\quad - \sum_{j=k}^{m-1} (z_j-\lambda_{j+1})\cdots(z_j-\lambda_m) C_{j-k}(z; z_k, \cdots, z_{j-1}; \lambda_{k+1}, \cdots, \lambda_j) \\
&= (z-\lambda_{k+1})\cdots(z-\lambda_m) \\
&\quad - \sum_{p=0}^{m-k-1} (z_{p+k}-\lambda_{p+k+1})\cdots(z_{p+k}-\lambda_m) C_p(z; z_k, \cdots, z_{p+k-1}; \lambda_{k+1}, \cdots, \lambda_{p+k}) \\
&= C_{m-k}(z; z_k, \cdots, z_{m-1}; \lambda_{k+1}, \cdots, \lambda_m).
\end{aligned}$$

Since (2.3) implies $\Delta^m C_m(z; z_0, \cdots, z_{m-1}; \lambda_1, \cdots, \lambda_m) = \Delta^m (z-\lambda_1)\cdots (z-\lambda_m) = 1$, the proof of (2.4) is complete.

For the proof of (2.5), note first that $C_1(z_0; z_0; \lambda_1) = (z_0 - \lambda_1) - (z_0-\lambda_1)C_0(z_0) = 0$. Let $m \ge 2$ be an integer and suppose

$$C_j(z_0; z_0, \cdots, z_{j-1}; \lambda_1, \cdots, \lambda_j) = 0 \quad \text{for } 1 \le j \le m-1.$$

Then

$$\begin{aligned}
C_m(z_0; z_0, \cdots, z_{m-1}; \lambda_1, \cdots, \lambda_m) \\
&= (z_0-\lambda_1)\cdots(z_0-\lambda_m) \\
&\quad - \sum_{j=0}^{m-1} (z_j-\lambda_{j+1})\cdots(z_j-\lambda_m) C_j(z_0; z_0, \cdots, z_{j-1}; \lambda_1, \cdots, \lambda_j) \\
&= (z_0-\lambda_1)\cdots(z_0-\lambda_m) - (z_0-\lambda_1)\cdots(z_0-\lambda_m) = 0.
\end{aligned}$$

The proofs of (2.7) and (2.9) are similar. Equation (2.6) follows from (2.4) and (2.5), together with the fact that $C_n(z)$ is a polynomial of degree n in z; (2.8) is the Newton series expansion of $C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$.

A convenient representation for the coefficient $C_n(\lambda_1; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$ in (2.8) is obtained by considering the infinite upper triangular matrices A and B defined as follows: for $0 \le j, k < \infty$ let

$$A_{jk} = \begin{cases} C_{k-j}(\lambda_{j+1}; z_j, \cdots, z_{k-1}; \lambda_{j+1}, \cdots, \lambda_k), & j < k, \\ 1, & j = k, \\ 0, & j > k, \end{cases}$$

(2.10)

$$B_{jk} = \begin{cases} (z_j - \lambda_{j+1})(z_j - \lambda_{j+2}) \cdots (z_j - \lambda_k), & j < k, \\ 1, & j = k, \\ 0, & j > k. \end{cases}$$

Thus A_{kn} is the kth coefficient of $C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$ in the expansion (2.8). It is easy to show that the matrices A and B are mutually reciprocal. For $j < n$, (2.3) implies

$$\sum_{k=j}^{n} A_{jk}B_{kn} = \sum_{k=j}^{n} C_{k-j}(\lambda_{j+1}; z_j, \cdots, z_{k-1}; \lambda_{j+1}, \cdots, \lambda_k)(z_k - \lambda_{k+1}) \cdots (z_k - \lambda_n)$$

$$= \sum_{m=0}^{n-j} C_m(\lambda_{j+1}; z_j, \cdots, z_{m+j-1}; \lambda_{j+1}, \cdots, \lambda_{m+j})(z_{m+j} - \lambda_{m+j+1}) \cdots (z_{m+j} - \lambda_n)$$

$$= (\lambda_{j+1} - \lambda_{j+1})(\lambda_{j+1} - \lambda_{j+2}) \cdots (\lambda_{j+1} - \lambda_n) = 0.$$

By applying (2.8) and then (2.6) and (2.4), we obtain

$$\sum_{k=j}^{n} B_{jk}A_{kn} = \sum_{k=j}^{n} (z_j - \lambda_{j+1}) \cdots (z_j - \lambda_k)C_{n-k}(\lambda_{k+1}; z_k, \cdots, z_{n-1}; \lambda_{k+1}, \cdots, \lambda_n)$$

$$= \sum_{m=0}^{n-j} (z_j - \lambda_{j+1}) \cdots (z_j - \lambda_{j+m})$$

$$\cdot\, C_{n-j-m}(\lambda_{m+j+1}; z_{m+j}, \cdots, z_{n-1}; \lambda_{m+j+1}, \cdots, \lambda_n)$$

$$= C_{n-j}(z_j; z_j, \cdots, z_{n-1}; \lambda_{j+1}, \cdots, \lambda_n) = 0.$$

Since $A_{nn} = B_{nn} = 1$, $0 \le n < \infty$, It follows that $AB = BA = I$, where I is the identity matrix.

Let m be a nonnegative integer and define the $(m+1)$ by $(m+1)$ matrices A^m and B^m by

$$A^m_{jk} = A_{jk}, \quad B^m_{jk} = B_{jk}, \quad 0 \le j,\ k \le m.$$

Arguing as above, it follows that A^m and B^m are inverses. Hence the entries in A^m can be determined by considering the cofactors in B^m. In particular,

(2.11)

$$A^m_{0,m} = C_m(\lambda_1; z_0, \cdots, z_{m-1}; \lambda_1, \cdots, \lambda_m)$$

$$= (-1)^m \mathrm{Det} \begin{bmatrix} (z_0 - \lambda_1) & (z_0 - \lambda_1)(z_0 - \lambda_2) & \cdots & [(z_0 - \lambda_1)(z_0 - \lambda_2) \cdots (z_0 - \lambda_m)] \\ 1 & (z_1 - \lambda_2) & \cdots & [(z_1 - \lambda_2)(z_1 - \lambda_3) \cdots (z_1 - \lambda_m)] \\ 0 & 1 & \cdots & [(z_2 - \lambda_3)(z_2 - \lambda_4) \cdots (z_2 - \lambda_m)] \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & (z_{m-1} - \lambda_m) \end{bmatrix}.$$

We will use (2.11) in establishing a relationship, in analogy to (2.9), between $C_n(\lambda_1; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$ and $B_n(0; z_0, \cdots, z_{n-1})$. Note first that if $z_0 \neq 0$, then (2.3) and (1.5) give $C_1(\lambda_1; z_0; \lambda_1) = -(z_0 - \lambda_1) = ((z_0 - \lambda_1)/z_0)(-z_0) = ((z_0 - \lambda_1)/z_0)B_1(0; z_0)$. Substituting this expression into (2.3), with $n = 2$, and using (1.5), we obtain

$$C_2(\lambda_1; z_0, z_1; \lambda_1, \lambda_2) = ((z_0 - \lambda_1)/z_0)B_2(0; z_0, z_1),$$

and similarly,

$$C_3(\lambda_1; z_0, z_1, z_2; \lambda_1, \lambda_2, \lambda_3) = ((z_0 - \lambda_1)/z_0)[B_3(0; z_0, z_1, z_2) - \lambda_2 B_2(0; z_0, z_1)].$$

In general, we can prove the following result.

Theorem 2. *If $n \geq 2$ and $z_0, z_1, \cdots, z_{n-1}$ are nonzero, then*

$$C_n(\lambda_1; z_0, z_1, \cdots, z_{n-1}; \lambda_1, \lambda_2, \cdots, \lambda_n)$$

(2.12)
$$= \left(\frac{z_0 - \lambda_1}{z_0}\right) \sum_{\substack{2 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-2}} (-1)^k \lambda_{j_1} \lambda_{j_2} \cdots \lambda_{j_k} \cdot B_{n-k}(0; z_0, z_1, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_2}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}).$$

The summation in (2.12) is taken over all possible configurations $\lambda_{p_1}\lambda_{p_2} \cdots \lambda_{p_k}$ such that $2 \leq p_1 < p_2 < \cdots < p_k \leq n-1$ and $0 \leq k \leq n-2$; if $k = 0$ we take $\lambda_{p_1} \cdots \lambda_{p_k} = 1$. The symbol $\widehat{z_{j_i}}$ means that the variable z_{j_i} has been removed; thus

$$B_{n-k}(0;\, z_0,\, z_1,\, \cdots,\, \widehat{z_{j_1}},\, \cdots,\, \widehat{z_{j_2}},\, \cdots,\, \widehat{z_{j_k}},\, \cdots,\, z_{n-1})$$

$$= B_{n-k}(0;\, z_0,\, z_1,\, \cdots,\, z_{j_1-1},\, z_{j_1+1},\, \cdots,\, z_{j_2-1},$$

$$z_{j_2+1},\, \cdots,\, z_{j_k-1},\, z_{j_k+1},\, \cdots,\, z_{n-1}),$$

To prove Theorem 2, we need the following technical lemma concerning the remainder polynomials.

Lemma 2. *If* $n \geq 1$, *then*

$$B_{n+1}(0;\, z_0,\, z_1,\, z_2,\, \cdots,\, z_n)$$

(2.13)

$$= z_0 B_n(0;\, z_0,\, z_2,\, z_3,\, \cdots,\, z_n) - z_0 B_n(0;\, z_1,\, z_2,\, \cdots,\, z_n).$$

If $n \geq 2$, $z_0 \neq 0$ *and* $z_1 \neq 0$, *then*

$$B_n(0;\, z_0,\, z_1,\, z_3,\, z_4,\, \cdots,\, z_n)$$

(2.14)

$$= B_n(0;\, z_0,\, z_2,\, z_3,\, \cdots,\, z_n) - (z_0/z_1) B_n(0;\, z_1,\, z_2,\, \cdots,\, z_n).$$

Proof. Consider (2.13). If $n = 1$, then

$$z_0 B_1(0;\, z_0) - z_0 B_1(0;\, z_1) = z_0(-z_0) - z_0(-z_1) = -z_0^2 + z_0 z_1 = B_2(0;\, z_0,\, z_1).$$

Let n be a positive integer and suppose that (2.13) holds for the integers k such that $1 \leq k \leq n$. Then by (1.5),

$$B_{n+2}(0;\, z_0,\, z_1,\, z_2,\, \cdots,\, z_{n+1}) = -\sum_{k=0}^{n+1} z_k^{n+2-k} B_k(0;\, z_0,\, z_1,\, \cdots,\, z_{k-1})$$

$$= -z_0^{n+2} - z_1^{n+1} B_1(0;\, z_0) - \sum_{k=2}^{n+1} z_k^{n+2-k} z_0 [B_{k-1}(0;\, z_0,\, z_2,\, \cdots,\, z_{k-1})$$

$$- B_{k-1}(0;\, z_1,\, z_2,\, \cdots,\, z_{k-1})]$$

$$= -z_0^{n+2} - z_0 \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0;\, z_0,\, z_2,\, \cdots,\, z_{k-1})$$

$$+ z_0 z_1^{n+1} + z_0 \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0;\, z_1,\, z_2,\, \cdots,\, z_{k-1})$$

$$= z_0 B_{n+1}(0;\, z_0,\, z_2,\, z_3,\, \cdots,\, z_{n+1}) - z_0 B_{n+1}(0;\, z_1,\, z_2,\, \cdots,\, z_{n+1}).$$

For the proof of (2.14) note first that $B_2(0; z_0, z_2) - (z_0/z_1)B_2(0; z_1, z_2) = - z_0^2 + z_0 z_2 - (z_0/z_1)(- z_1^2 + z_1 z_2) = - z_0^2 + z_0 z_2 + z_0 z_1 - z_0 z_2 = - z_0^2 + z_0 z_1 = B_2(0; z_0, z_1)$. If (2.14) holds for the integers k such that $2 \leq k \leq n$, then

$$\begin{aligned}
&B_{n+1}(0; z_0, z_1, z_3, \cdots, z_{n+1}) \\
&\quad = - z_0^{n+1} - z_1^n(- z_0) - \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_1, z_3, \cdots, z_{k-1}) \\
&\quad = - z_0^{n+1} + z_0 z_1^n - \sum_{k=3}^{n+1} z_k^{n+2-k} \Big[B_{k-1}(0; z_0, z_2, z_3, \cdots, z_{k-1}) \\
&\qquad\qquad - \frac{z_0}{z_1} B_{k-1}(0; z_1, z_2, \cdots, z_{k-1}) \Big] \\
&\quad = - z_0^{n+1} - z_2^n B_1(0; z_0) - \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_2, \cdots, z_{k-1}) \\
&\qquad + z_0 z_1^n + z_2^n B_1(0; z_0) + \frac{z_0}{z_1} \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \cdots, z_{k-1}) \\
&\quad = - z_0^{n+1} - \sum_{k=2}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_0, z_2, z_3, \cdots, z_{k-1}) \\
&\qquad + \frac{z_0}{z_1} \left[z_1^{n+1} + z_2^n B_1(0; z_0) \frac{z_1}{z_0} + \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \cdots, z_{k-1}) \right] \\
&\quad = B_{n+1}(0; z_0, z_2, z_3, \cdots, z_{n+1}) \\
&\qquad + \frac{z_0}{z_1} \left[z_1^{n+1} + z_2^n B_1(0; z_1) + \sum_{k=3}^{n+1} z_k^{n+2-k} B_{k-1}(0; z_1, z_2, \cdots, z_{k-1}) \right] \\
&\quad = B_{n+1}(0; z_0, z_2, \cdots, z_{n+1}) - (z_0/z_1) B_{n+1}(0; z_1, z_2, \cdots, z_{n+1}),
\end{aligned}$$

and this completes the proof.

Proof of Theorem 2. The proof is by induction; the case $n = 2$ has been established. Suppose now that $n > 2$ and (2.12) holds for the integers k such that $2 \leq k \leq n - 1$. Taking $m = n$ in (2.11) and expanding the determinant by the first column, we have

$$\begin{aligned}
&C_n(\lambda_1; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) \\
&\quad = (-1)^n[(-1)^{n-1}(z_0 - \lambda_1) C_{n-1}(\lambda_2; z_1, z_2, \cdots, z_{n-1}; \lambda_2, \lambda_3, \cdots, \lambda_n)] \\
&\qquad - (-1)^n[(-1)^{n-1}(z_0 - \lambda_2) C_{n-1}(\lambda_1; z_0, z_2, \cdots, z_{n-1}; \lambda_1, \lambda_3, \cdots, \lambda_n)].
\end{aligned}$$

By the induction hypothesis,

$$C_n(\lambda_1; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$$

$$= -(z_0 - \lambda_1)\left(\frac{z_1 - \lambda_2}{z_1}\right) \sum_{\substack{3 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-3}} (-1)^k \lambda_{j_1} \cdots \lambda_{j_k} B_{n-k-1}(0; z_1, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1})$$

$$+ (z_0 - \lambda_2)\left(\frac{z_0 - \lambda_1}{z_0}\right) \sum_{\substack{3 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-3}} (-1)^k \lambda_{j_1} \cdots \lambda_{j_k} B_{n-k-1}(0; z_0, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}).$$

By (2.13) and (2.14), the coefficient of $((z_0 - \lambda_1)/z_0)(-1)^k \lambda_{j_1} \cdots \lambda_{j_k}$ is given by

$$\begin{aligned}
&-((z_1 - \lambda_2)/z_1) z_0 B_{n-k-1}(0; z_1, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&\qquad + (z_0 - \lambda_2) B_{n-k-1}(0; z_0, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&= z_0 B_{n-k-1}(0; z_0, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&\quad - z_0 B_{n-k-1}(0; z_1, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_2}}, \cdots, z_{n-1}) \\
&\quad - \lambda_2 [B_{n-k-1}(0; z_0, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&\qquad - (z_0/z_1) B_{n-k-1}(0; z_1, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1})] \\
&= B_{n-k}(0; z_0, z_1, z_2, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&\quad - \lambda_2 B_{n-k-1}(0; z_0, z_1, \widehat{z_2}, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
&C_n(\lambda_1; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n) \\
&= \left(\frac{z_0 - \lambda_1}{z_0}\right) \sum_{\substack{3 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-3}} (-1)^k \lambda_{j_1} \cdots \lambda_{j_k} \cdot B_{n-k}(0; z_0, z_1, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}) \\
&\quad + \left(\frac{z_0 - \lambda_1}{z_0}\right) \sum_{\substack{3 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-3}} (-1)^{k+1} \lambda_2 \lambda_{j_1} \cdots \lambda_{j_k} \cdot B_{n-k-1}(0; z_0, z_1, \widehat{z_2}, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1})
\end{aligned}$$

$$= \left(\frac{z_0 - \lambda_1}{z_0}\right) \sum_{\substack{2 \le j_1 < \cdots < j_k \le n-1 \\ 0 \le k \le n-2}} (-1)^k \lambda_{j_1} \cdots \lambda_{j_k} \cdot B_{n-k}(0; z_0, z_1, \cdots, \widehat{z_{j_1}}, \cdots, \widehat{z_{j_k}}, \cdots, z_{n-1}),$$

and this completes the proof.

We conclude this section now with a lemma to be used in § 3 below.

Lemma 3. *Let $n \ge 1$ and suppose that the complex numbers $z_0, z_1, \cdots, z_{n-1}$ lie in the disc $|z| \le W$. Then*

$$|B_n(0; z_0, \cdots, z_{n-1})| \le 1. \tag{2.15}$$

Moreover, there exist complex numbers $\zeta_0, \zeta_1, \cdots, \zeta_{n-1}$ on the circle $|z| = W$ such that

$$|B_n(0; \zeta_0, \zeta_1, \cdots, \zeta_{n-1})| = W^n H_n. \tag{2.16}$$

Proof. Let $w_k = z_k/W$, $0 \le k \le n-1$. Then $|w_k| \le 1$, $0 \le k \le n-1$, and (2.7), (2.9) and (1.6) imply $|B_n(0; z_0, \cdots, z_{n-1})| = W^n |B_n(0; w_0, \cdots, w_{n-1})| \le W^n H_n \le 1$. For (2.16), the maximum principle permits us to choose points w_0', $w_1', \cdots, w_{n-1}'$ on the circle $|z| = 1$ such that $|B_n(0; w_0', \cdots, w_{n-1}')| = H_n$. If $\zeta_k = W w_k'$, $0 \le k \le n-1$, then $|B_n(0; \zeta_0, \cdots, \zeta_{n-1})| = |W^n B_n(0; w_0', \cdots, w_{n-1}')| = W^n H_n$, and this completes the proof.

3. **Matrix transformations on the space $\mathfrak{A}_r$.** If $r > 0$, we denote by $\mathfrak{A}_r$ the complex vector space of functions analytic in the disc $D_r = \{z: |z| < r\}$; $\mathfrak{A}_r$, given the topology of uniform convergence on compact subsets of D_r, is known to be a Fréchet-Montel space [8].

Let $\lambda \in c_0$ have terms in D_r and define $\sigma_0(z) = 1$, $\sigma_n(z) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_n)$, $n = 1, 2, 3, \cdots$. Then $\{\sigma_n\}_{n=0}^{\infty}$ is a basis for $\mathfrak{A}_r$ [10]; in fact, $f = \sum_{n=0}^{\infty} a_n \sigma_n$ is the Newton series expansion for $f \in \mathfrak{A}_r$. It is not difficult to show that $f = \sum_{n=0}^{\infty} a_n \sigma_n$ belongs to $\mathfrak{A}_r$ if and only if $\limsup_{n\to\infty} |a_n|^{1/n} \le r^{-1}$. Thus we may identify $\mathfrak{A}_r$ with the space of complex sequences $x = (x_0, x_1, \cdots)$ such that $\limsup_{n\to\infty} |x_n|^{1/n} \le 1/r$.

Let M be an infinite complex matrix and let $f = (f_0, f_1, \cdots) \in \mathfrak{A}_r$. Then Mf is the sequence whose nth term is given by $(Mf)_n = \sum_{k=0}^{\infty} M_{nk} f_k$. We say that M maps $\mathfrak{A}_r$ to $\mathfrak{A}_r$ provided that $f \in \mathfrak{A}_r$ implies $Mf \in \mathfrak{A}_r$. The following theorem of M. M. Dragilev ([5], [6]) characterizes a large class of matrix transformations on $\mathfrak{A}_r$.

Theorem 3. *Let M be an infinite upper triangular matrix such that $M_{kk} = 1$, $0 \le k < \infty$, let N denote the unique upper triangular inverse of M, and suppose*

$R > 0$. A necessary and sufficient condition that either M or N map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for each $r > R$ is that

$$\limsup_{n\to\infty} \left[\max_{0\le j\le n} |M_{jn}| R^{j-n}\right]^{1/n} \le 1 \tag{3.1}$$

and

$$\limsup_{n\to\infty} \left[\max_{0\le j\le n} |N_{jn}| R^{j-n}\right]^{1/n} \le 1. \tag{3.2}$$

Furthermore, it is not difficult to show that either M and N simultaneously map $\mathfrak{A}_r$ one-to-one onto itself or neither does.

Lemma 4. *Suppose $R > 0$ and let $\{\lambda_k\}_{k=1}^\infty$ and $\{z_k\}_{k=0}^\infty$ be sequences in D_R such that $\lambda_k \to 0$ and $|z_k| \le WR$, $0 \le k < \infty$. Then the matrices A and B of* (2.10) *satisfy* (3.1) *and* (3.2).

Proof. Consider the matrix B. If $r_n = |\lambda_n|$, $1 \le n < \infty$, and $j < k$, then

$$\begin{aligned} |B_{jk}| &= |(z_j - \lambda_{j+1})(z_j - \lambda_{j+2}) \cdots (z_j - \lambda_k)| \\ &\le (WR + r_{j+1})(WR + r_{j+2}) \cdots (WR + r_k) \\ &= (WR)^{k-j}[(1 + r_{j+1}/WR) \cdots (1 + r_k/WR)] \\ &\le (WR)^{k-j}(1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k), \end{aligned}$$

where $\epsilon_n = r_n/(WR)$. Now $\epsilon_n \to 0$ and therefore

$$\lim_{n\to\infty} [(1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_n)]^{1/n} = 1.$$

Since $W < 1$, we have $|B_{jk}|R^{j-k} \le W^{k-j}(1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k) < (1 + \epsilon_1)(1 + \epsilon_2) \cdots (1 + \epsilon_k)$ and it follows that

$$\limsup_{k\to\infty} \left[\max_{0\le j\le k} |B_{jk}| R^{j-k}\right]^{1/k} \le 1.$$

For the matrix A, (2.7) implies

$$|A_{jk}|R^{j-k} = |C_{k-j}(\lambda_{j+1}/R;\ z_j/R, \cdots, z_{k-1}/R;\ \lambda_{j+1}/R, \cdots, \lambda_k/R)|,$$

for $j < k$. By the maximum principle, there exist points $w_j, w_{j+1}, \cdots, w_k$ on the circle $|z| = W$ such that $|A_{jk}|R^{j-k} \le |C_{k-j}(\mu_{j+1};\ w_j, \cdots, w_{k-1};\ \mu_{j+1}, \cdots, \mu_k)|$, where $\mu_i = \lambda_i/R$, $1 \le i < \infty$. By (2.12) and Lemma 3,

$$|A_{jk}|R^{j-k} \le \left|\frac{w_j - \mu_{j+1}}{w_j}\right| \sum_{\substack{j+2\le p_1<\cdots<p_n\le k-1 \\ 0\le n\le k-j-2}} |\mu_{p_1}||\mu_{p_2}|\cdots|\mu_{p_n}|$$

$$\le \frac{W+1}{W}(1+|\mu_1|)(1+|\mu_2|)\cdots(1+|\mu_{k-1}|).$$

Since $|\mu_n| \to 0$, it follows that $\lim\sup_{k\to\infty}[\max_{0\le j\le k}|A_{jk}|R^{j-k}]^{1/k} \le 1$, and this completes the proof.

Theorem 4. *Let f be analytic in D, let $0 < R < 1$, and suppose that $\{z_k\}_{k=0}^{\infty}$ and $\{\lambda_k\}_{k=1}^{\infty}$ are sequences in D such that $|z_k| \le WR$, $0 \le k < \infty$, and $\lambda_k \to 0$. Then*

$$f(z) = \sum_{k=0}^{\infty} \Delta^k f(z_k) C_k(z; z_0, \cdots, z_{k-1}; \lambda_1, \cdots, \lambda_k), \tag{3.3}$$

with uniform convergence on compact subsets of D.

Proof. By Theorem 3 and Lemma 4, the matrices A and B map $\mathfrak{A}_r$ one-to-one and onto $\mathfrak{A}_r$ for each $r > R$, where $0 < R < 1$. Thus A and B map $\mathfrak{A}_1$ one-to-one onto itself. By a theorem of Köthe and Toeplitz [9], A and B are weakly continuous and, hence, [11, p. 158] are continuous. It follows that $\{A(\sigma_n)\}_{n=0}^{\infty}$ is a basis for $\mathfrak{A}_1$, since $\{\sigma_n\}_{n=0}^{\infty}$ is a basis. But (2.8) implies that $A(\sigma_n)(z) = C_n(z; z_0, \cdots, z_{n-1}; \lambda_1, \cdots, \lambda_n)$ $(0 \le n < \infty)$, and therefore functions $f \in \mathfrak{A}_1$ admit expansions $f(z) = \sum_{k=0}^{\infty} \alpha_k C_k(z; z_0, \cdots, z_{k-1}; \lambda_1, \cdots, \lambda_k)$ with uniform convergence on compact subsets of D. The linear functionals $g \to \Delta^j g(z_j)$, $0 \le j < \infty$, are readily shown to be continuous. This fact, together with (2.6) implies $\alpha_k = \Delta^k f(z_k)$, $0 \le k < \infty$, which completes the proof.

Since $R < 1$ is arbitrary in Theorem 4, (3.3) implies that $C_\lambda \ge W$ for all $\lambda \in c_0$, which proves the first part of Theorem 1. To complete the proof of Theorem 1, we will prove that for each $\lambda \in l_1$, there exists an *extremal* function $F \in \mathfrak{A}_1$ such that each of $\Delta_\lambda^k F(z)$ has a zero in $|z| \le W$ but $F \not\equiv 0$. With the expansion (3.3), this will imply $C_\lambda \le W$ for all $\lambda \in l_1$.

Thus suppose $\lambda \in l_1$ and let $r_k = |\lambda_k|$, $1 \le k < \infty$. Then the sequence $s_n = (1 + r_1)(1 + r_2)\cdots(1 + r_n)$ converges [7, p. 223] and so there exists a constant $K > 1$ such that $(1 + r_1)(1 + r_2)\cdots(1 + r_n) < K$ for all n.

Let N be a positive integer such that $r_n < W/2$, for $n \ge N$, and such that $\sum_{n=N}^{\infty} r_n < \beta/(\beta + 2)$ (see (1.7)). If $n \ge N$, there exist, by Lemma 3, points $z_0^{(n)}$, $z_1^{(n)}, \cdots, z_{n-1}^{(n)}$ on $|z| = W$ such that

$$|B_{n-N}(0; z_N^{(n)}, z_{N+1}^{(n)}, \cdots, z_{n-1}^{(n)})| = W^{n-H} H_{n-N}. \tag{3.4}$$

For $n \geq N$, define $F_n(z) = C_n(z; z_0^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_1, \ldots, \lambda_n)$. By (2.8), we have

$$F_n(z) = \sum_{k=0}^{n} C_{n-k}(\lambda_{k+1}; z_k^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_{k+1}, \ldots, \lambda_n)\sigma_k(z). \tag{3.5}$$

As in the proof of Lemma 4,

$$|C_{n-k}(\lambda_{k+1}; z_k^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_{k+1}, \ldots, \lambda_n)|$$

$$\leq ((W+1)/W)(1+r_1)(1+r_2)\cdots(1+r_{n-1}) \leq ((W+1)/W)K,$$

for $0 \leq k \leq n$. Therefore for $n \geq N$ and $z \in D$, $|F_n(z)| \leq ((W+1)/W)K \sum_{k=0}^{n} |\sigma_k(z)|$. Noting that $\{\sigma_k\}$ is a basis, the sequence $\{F_n\}_{n=N}^{\infty}$ is uniformly bounded on compact subsets of D. Therefore, there exists a set S of positive integers such that the sequence $\{F_n\}_{n \in S}$ converges uniformly on compact subsets of D to a function $F \in \mathfrak{A}_1$. For each nonnegative integer k, the sequence $\{\Delta^k F_n\}_{n \in S}$ converges uniformly on compact subsets to $\Delta^k F$, by continuity of the map $f \to \Delta^k f$. Since $\Delta^k F_n(z_k^{(n)}) = 0$, $n \in S$, uniform convergence implies that $\Delta^k F$ has a zero on the circle $|z| = W$.

To complete the construction, there remains to show $F \not\equiv 0$. Since $F(z) = F(\lambda_1) + \sum_{k=1}^{\infty} \Delta^k F(\lambda_{k+1})\sigma_k(z)$, it suffices to show that $\Delta^N F(\lambda_{N+1}) \neq 0$. By (3.5).

$$\Delta^N F(\lambda_{N+1}) = \lim_{n \in S} C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_{N+1}, \ldots, \lambda_n).$$

For each $n \geq N+2$, (3.4), (1.7) and (2.12) imply that

$$|C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_{N+1}, \ldots, \lambda_n)|$$

$$\geq \left| \frac{z_N^{(n)} - \lambda_{N+1}}{z_N^{(n)}} \right| \left\{ \beta - \sum_{\substack{N+2 \leq p_1 < \cdots < p_m \leq n-1 \\ 1 \leq m \leq n-N-2}} r_{p_1} r_{p_2} \cdots r_{p_m} \right\}.$$

Writing $\tau_n = r_{N+2} + r_{N+3} + \cdots + r_{n-1}$, the previous inequality implies

$$|C_{n-N}(\lambda_{N+1}; z_N^{(n)}, \ldots, z_{n-1}^{(n)}; \lambda_{N+1}, \ldots, \lambda_n)|$$

$$\geq \frac{W - W/2}{W} \{\beta - \tau_n - \tau_n^2 - \cdots - \tau_n^{n-N-2}\}$$

$$\geq \frac{1}{2} \left\{ \beta - \sum_{j=1}^{\infty} \left(\sum_{n=N}^{\infty} r_n \right)^j \right\} \geq \frac{1}{2} \left\{ \beta - \sum_{j=1}^{\infty} \left(\frac{\beta}{2+\beta} \right)^j \right\} = \frac{\beta}{4}$$

for each $n \in S$ with $n \geq N+2$, and it follow that $|\Delta^N F(\lambda_{N+1})| \geq \beta/4 > 0$.

Remark. Fix $\rho > 0$ and define $C_{\lambda,\rho}$ to be the supremum of numbers $c > 0$

such that if $f \in \mathfrak{A}_\rho$ and each of $\Delta_\lambda^k f(z)$, $0 \le k < \infty$, has a zero in $|z| \le c$, then $f \equiv 0$. By taking $0 < R < \rho$ in Theorem 4, we see that $C_{\lambda,\rho} \ge \rho W$ for all $\lambda \in c_0$. If $\lambda \in l_1$, let $\mu = \rho^{-1}\lambda = (\rho^{-1}\lambda_1, \rho^{-1}\lambda_2, \cdots)$ and construct the extremal function $F \in \mathfrak{A}_1$ with respect to the sequence μ. Define $G(z) = F(z/\rho)$. Then $G \in \mathfrak{A}_\rho$, $G \not\equiv 0$, and each of $\Delta_\lambda^k G(z)$ has a zero on $|z| = \rho W$. It follows that $C_{\lambda,\rho} = \rho W$ for $\lambda \in l_1$.

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